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Linear Functionals Defined on Various Spaces of Continuous Functions on R

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1. INTRODUCTION

More than twenty years ago one of the authors first became acquainted with the very elegant Hildebrandt–Schoenberg [26] proof of the Riesz representation theorem for bounded linear functionals on the space $C\,[0, 1]$ when, as a student of Professor G. G. Lorentz, he worked over his excellent book [28] on Bernstein polynomials. For this proof uses the fact that these polynomials tend uniformly to the function in question on [0, 1] (see, e.g., I. P. Natanson [31, p. 241], or T. H. Hildebrandt [25, p. 84]) plus the theorem of Helly-Bray. At that time the question arose whether one could prove the corresponding version for the infinite interval **R** by finding a suitable approximation process (taking the place of the Bernstein polynomials) which converges uniformly on the whole of **R**, instead of following the more modern procedure of deducing Riesz's theorem on $C_0(\mathbf{R})$ from abstract results on measure and integration on locally compact spaces (compare [35, p. 131; 23, p. 177] or [36, p. 318]). It turns out that interpolating splines of order 1 form a suitable approximation process for the above purpose. This leads to Theorem 3 of Section 4.

Now, for functions defined on the whole of **R**, namely the locally compact situation, a number of spaces have to be distinguished, to which we turn next.

Let $B(\mathbf{R})$ denote the vector space of all bounded real-valued functions defined on the real axis **R**, and $C(\mathbf{R})$ the space of all continuous real-valued functions defined on **R**. Let $C_B(\mathbf{R})$ denote the subset of $C(\mathbf{R})$ consisting of all bounded functions, $C_0(\mathbf{R})$ the set of those $f \in C(\mathbf{R})$ for which $\lim_{|x|\to\infty} f(x) = 0$, and let $C_{00}(\mathbf{R})$ be the set of all $f \in C_0(\mathbf{R})$ having compact support. Apart from

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$C(\mathbf{R})$ and $C_{00}(\mathbf{R})$ all of these spaces are Banach spaces with respect to the norm

$$\|f\|_C = \sup_{x \in \mathbf{R}} |f(x)|;$$

$C_{00}(\mathbf{R})$ is a normed space under the latter norm, its completion being $C_0(\mathbf{R})$. Also

$$C_{00}(\mathbf{R}) \subset C_0(\mathbf{R}) \subset C_B(\mathbf{R}) \subset \begin{cases} C(\mathbf{R}) \\ B(\mathbf{R}) \end{cases}.$$

A linear functional L on any of the above spaces is said to be bounded if there is a constant $M > 0$ such that $|L(f)| \leqslant M \|f\|_C$. Moreover, let $BV(\mathbf{R})$ denote the vector space of real-valued functions α which are of bounded variation on $\mathbf{R}$, i.e., for which the total variation $[\mathrm{Var}\ \alpha]_{\mathbf{R}}$ is finite, and which are normalized by $\alpha(-\infty) = \alpha(-\infty - 0) = 0$, $\alpha(+\infty) = \alpha(+\infty - 0)$, and $\alpha(x) = [\alpha(x+0) + \alpha(x-0)]/2$ for $-\infty < x < \infty$.

The main purpose of this paper is to classify the various integral representations for linear functionals defined not only on the space $C_0(\mathbf{R})$, but also on $C(\mathbf{R})$, $C_{00}(\mathbf{R})$, $C_B(\mathbf{R})$, and $B(\mathbf{R})$, and to investigate the limitations inherent in the possible integral representations for these various spaces. Thus whereas linear functionals on $C(\mathbf{R})$, $C_0(\mathbf{R})$ and $C_{00}(\mathbf{R})$ are expressible as tangible Riemann–Stieltjes integrals over $\mathbf{R}$ with respect to functions of bounded variation, those on $C_B(\mathbf{R})$ or $B(\mathbf{R})$ cannot be represented in this form but as integrals with respect to "finitely additive measures," which, to quote R. E. Edwards [12, p. 213], "can exhibit behavior that is almost barbaric."

In connection with the representation theorem for the space $C(\mathbf{R})$, also the associated Hamburger moment problem will be considered. Of the three or so ways in solving the latter, one is the original (lengthy) approach of H. L. Hamburger [19] using continued fractions (a method introduced by Stieltjes [39] to solve the moment problem named after him and also used by later authors, e.g., Shohat and Tamarkin [37], and Achieser [1]). Another method makes use of the connection between moment problems and quadratic forms (e.g., Achieser and Krein [2], D. V. Widder [46], and I. P. Natanson [31]). The third consists in first establishing a Riesz theorem for positive linear functionals defined on $C(\mathbf{R})$, and then deducing the solution to Hamburger's problem as a simple corollary. This approach is due to M. G. Krein [2, p. 137ff]; see also R. Arens [5] and the recent book by M. Cotlar and R. Cignoli [10, p. 157]. The prototype result here is that Riesz's theorem for $C\,[0, 1]$ can be used to solve Hausdorff's moment problem on $[0, 1]$ (and conversely). To establish a representation theorem for linear functionals over the space $C(\mathbf{R})$, the functions of which may be unbounded, Krein restricts these to so-called *normal* functions, i.e., functions $f \in C(\mathbf{R})$ for which there exists some positive $\omega_f \in C(\mathbf{R})$ such that $f(x) = o(\omega_f(x))$ as $|x| \to \infty$.

Our procedure will be to try to simplify Krein's method (he himself refers to it as a development of an idea of M. Riesz [32]) by replacing his extension theorem on linear functionals by another result inspired by one of R. Arens [5] (which will also be needed to prove Theorem 4), and by emphasizing the approximation by splines of order zero in the main part of the proof. This will be Theorem 1, and the application to Hamburger's problem is Theorem 2, both found in Section 3.

Although there is an enormous literature on integral representations, especially for $C(X)$, X a compact space (for which there also exists a recent survey article by J. Batt [7]), the literature for the locally compact case, particularly for $\mathbf{R}$ treated here, is more modest. This is one justification for this paper. The other is that whereas most of the papers that have appeared in the past three decades are based on an "abstract" approach, the present one is very "concrete", and so may be followed, e.g., by senior undergraduates. We have in fact tried to preserve the simplicity and elegance of the Hildebrandt–Schoenberg approach, interpolating splines playing here the role of Bernstein polynomials. Although some of the lemmas and theorems below may not be new, we hope to have presented a few new proofs, emphasis being placed on systematic presentation. Although we have not attempted to write a survey article covering $\mathbf{R}$, we have tried to mention the relevant papers we saw. In this connection see e.g., N. Bourbaki [9, p. 113–126].

2. Preliminary Results on Splines

For the following we need:

Definition 1. Setting

$$s(u) = \begin{cases} 0 \ , & \text{for} \ |u| \geqslant 1, \\ u + 1 \ , & \text{for} \ -1 < u \leqslant 0, \\ -u + 1, & \text{for} \ 0 < u < 1, \end{cases}$$

define

$$s_{k,n}(u) = s(nu - k) \qquad (n \in \mathbf{N}, k \in \mathbf{Z}).$$

Lemma 1 (I. J. Schoenberg). *For $f \in C_0(\mathbf{R})$, the interpolating splines*

$$(S_n f)(u) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) s_{k,n}(u),$$

have the property that

$$\lim_{n\to\infty} (S_n f)(u) = f(u)$$

uniformly on $\mathbf{R}$. Moreover, $f \in C_0(\mathbf{R})$ implies $S_n(f; u) \in C_0(\mathbf{R})$, $n \in \mathbf{N}$.

Proof. Let both $n \in \mathbf{N}$, and $u \in \mathbf{R}$ be arbitrary but fixed. Then there is $j \in \mathbf{Z}$ such that $j/n < u \leqslant (j+1)/n$. Since $s_{k,n}(u) = 0$ for all $k \in \mathbf{Z}$ with $k \neq j, j+1$, one has

$$\begin{aligned}
&|(S_n f)(u) - f(u)| \\
&= \Big| \sum_{k=-\infty}^{\infty} f\Big(\frac{k}{n}\Big) s_{k,n}(u) - f(u) \Big| \\
&= \Big| f\Big(\frac{j}{n}\Big)(j+1-nu) + f\Big(\frac{j+1}{n}\Big)(nu-j) - f(u) \Big| \\
&\leqslant \Big| f\Big(\frac{j+1}{n}\Big) - f(u) \Big| \, |nu-j| + \Big| f\Big(\frac{j}{n}\Big) - f(u) \Big| \, |j+1-nu| \\
&\leqslant \Big| f\Big(\frac{j+1}{n}\Big) - f(u) \Big| + \Big| f\Big(\frac{j}{n}\Big) - f(u) \Big| \\
&\leqslant 2 \sup_{|x-u| \leqslant 1/n} |f(x) - f(u)| ,
\end{aligned}$$

which tends to zero as $n \to \infty$, uniformly on $\mathbf{R}$, since $f \in C_0(\mathbf{R})$. The rest of the proof is clear.

We also need the following two simple "approximation" results for splines of order zero. Here χ_E denotes the characteristic function of $E \subset \mathbf{R}$, i.e., $\chi_E(u) = 1$ for $u \in E$, $= 0$ for $u \notin E$.

LEMMA 2. *If $f \in C[a, b]$, $\Delta : a = u_0 < u_1 < \cdots < u_p = b$, and*

$$X_\Delta(f; u) = \sum_{k=1}^{p} f(u_k) \chi_{(u_{k-1}, u_k]}(u),$$

then

$$\lim_{\|\Delta\| \to 0} X_\Delta(f; u) = f(u)$$

uniformly on $(a, b]$. ($\|\Delta\| := \max_k (x_{k+1} - x_k)$).

Proof. For any $u \in (a, b]$, there is j, $0 \leqslant j \leqslant p-1$, with $u_j < u \leqslant u_{j+1}$. Hence

$$|f(u) - X_\Delta(f; u)| = |f(u) - f(u_{j+1}) \chi_{(u_j, u_{j+1}]}(u)| ,$$

since the other $\chi_{(u_k, u_{k+1}]}$ vanish for $k \neq j$. The result then follows.

LEMMA 3. *Let $f \in B(\mathbf{R})$, $c = \inf\{f(x); x \in \mathbf{R}\}$, $d = \sup\{f(x); x \in \mathbf{R}\}$, and let (C, D) be any interval containing $[c, d]$. Let $\Delta' : C = \beta_0 < \beta_1 < \cdots < \beta_p = D$. Then*

$$\lim_{\|\Delta'\| \to 0} \sum_{k=1}^{p} \beta_{k-1} \chi_{f^{-1}(\beta_{k-1}, \beta_k]}(u) = f(u)$$

uniformly on $\mathbf{R}$.

Proof. Let $u \in \mathbf{R}$ be arbitrary but fixed. Then there is j, $1 \leqslant j \leqslant p$, such that $\beta_{j-1} < f(u) \leqslant \beta_j$, implying that $u \in f^{-1}(\beta_{j-1}, \beta_j] := \{u \in \mathbf{R}; f(u) \in (\beta_{j-1}, \beta_j]\}$. Hence

$$\left| f(u) - \sum_{k=1}^{p} \beta_{k-1} \chi_{f^{-1}(\beta_{k-1}, \beta_k]}(u) \right| = | f(u) - \beta_{j-1} | \leqslant \beta_j - \beta_{j-1} \leqslant \| \Delta' \| < \epsilon$$

for $\| \Delta' \| < \delta(\epsilon) := \epsilon$. This establishes the result.

3. The Riesz Representation Theorem for Positive Linear Functionals on $C(\mathbf{R})$ and the Hamburger Moment Problem

As noted in the introduction, to cover the case of positive linear functionals on $C(\mathbf{R})$, the functions of which are not necessarily bounded, one must restrict the class of functions admitted.

Definition 2. Let $\mathfrak{E}$ be a linear manifold in $C(\mathbf{R})$. A function $f \in \mathfrak{E}$ is said to be *o-normal* (relative to $\mathfrak{E}$) if there exists (at least one) $\omega_f \in \mathfrak{E}$ such that

(i) $\omega_f(x) > 0$ for $| x |$ sufficiently large,

(ii) $f(x) = o(\omega_f(x))$ $(| X | \to \infty)$.

$f \in \mathfrak{E}$ is said to be *O-normal* if the above holds with (ii) replaced by

(ii)′ $f(x) = O(\omega_f(x))$ $(| x | \to \infty)$.

Obviously any function that is o-normal is also O-normal.

As examples of linear manifolds $\mathfrak{E}$ in $C(\mathbf{R})$ which consist *only* of o-normal functions let us mention the space $C(\mathbf{R})$ itself (for $f \in C(\mathbf{R})$ choose $\omega_f(x) = | x |(1 + | f(x)|) \in C(\mathbf{R})$), $C_0(\mathbf{R})$ (for $f \in C_0(\mathbf{R})$ choose $\omega_f(x) = (| f(x)|)^{1/2} + \exp(-x^2) \in C_0(\mathbf{R})$), as well as the set P of all algebraic polynomials $p_l(x) = \sum_{k=0}^{l} a_k x^k$ (for $p_l \in P$ choose $\omega_{p_l}(x) = x^{2l}$).

We need a lemma on extension of positive linear functionals; it is inspired by one due to R. Arens [5].

Lemma 4. *Let L be a positive linear functional defined on a linear manifold $\mathfrak{E} \subset C(\mathbf{R})$ containing $f_0(x) := 1$ $(x \in \mathbf{R})$. Assume further that each function of $\mathfrak{E}$ is O-normal. Then L can be extended to a positive linear functional L^* defined on $\mathfrak{E}^* := \mathfrak{E} +$ span $X(C_B)$, where*

$$X(C_B) = \{ g \cdot \chi_A \,;\, g \in C_B(\mathbf{R}),\, A \in \mathfrak{P}(\mathbf{R})\}.$$

($\mathfrak{P}(\mathbf{R})$ is the power set of $\mathbf{R}$, i.e., the set of all subsets of $\mathbf{R}$).

Proof. Suppose L' is a positive linear extension of L to some subspace $\mathfrak{E}'$, $\mathfrak{E} \subset \mathfrak{E}' \subset \mathfrak{E}^*$, and let $f \in \mathfrak{E}^* \backslash \mathfrak{E}'$. Defining

$$L_*'(f) = \sup_{g \leqslant f, g \in \mathfrak{E}'} L'(g), \qquad L'^*(f) = \inf_{g \geqslant f, g \in \mathfrak{E}'} L'(g), \tag{3.1}$$

we have

$$L_*'(f) \leqslant L'^*(f). \tag{3.2}$$

We show that the extensions $L_*'(f)$, $L'^*(f)$ are well-defined. Since f has the representation

$$f = g_0 + \sum_{k=1}^{m} \alpha_k g_k \chi_{A_k}$$

with $A_k \in \mathfrak{P}(\mathbf{R})$, $\alpha_k \in \mathbf{R}$, $g_0 \in \mathfrak{E}$, $g_k \in C_B(\mathbf{R})$, there exists an $\omega_{g_0} \in \mathfrak{E}$ and a compact set K such that $|g_0| \leqslant \omega_{g_0}$ for all $x \in \mathbf{R}\backslash K$, g_0 being O-normal. Moreover, each $g_k \in C_B(\mathbf{R})$, and so there exists a constant $M_1 > 0$ with $|\sum_{k=1}^{m} \alpha_k g_k \chi_{A_k}| \leqslant M_1$ for all $x \in \mathbf{R}\backslash K$. This yields

$$|f(x)| \leqslant \omega_{g_0}(x) + M_1 \qquad (x \in \mathbf{R}\backslash K). \tag{3.3}$$

Since f, as a sum of bounded functions in K, is also bounded in K, there is a constant $M_2 > 0$ such that $|f| \leqslant M_2$ for all $x \in K$. Together with (3.3) this implies that

$$|f(x)| \leqslant \omega_{g_0}(x) + M_1 + M_2 \qquad (\text{all } x \in \mathbf{R}),$$

where $M_i \in \mathfrak{E}$, $i = 1, 2$, since $f_0 \in \mathfrak{E}$. Thus,

$$L'^*(f) \leqslant L(\omega_{g_0}) + L(M_1) + L(M_2) < \infty,$$

$$L_*'(f) = -L'^*(-f) > -\infty,$$

so that $L'^*(f)$, $L_*'(f)$ are well-defined. Hence there exists a constant $\lambda \in \mathbf{R}$ such that $L_*'(f) \leqslant \lambda \leqslant L'^*(f)$. Setting for arbitrary $\eta \in \mathbf{R}$, $g \in \mathfrak{E}'$, $L''(\eta f + g) = \eta\lambda + L'(g)$, we obtain an extension L'' of L' on $\mathfrak{E}'$ to a larger subspace $\mathfrak{E}'' = \operatorname{span} f + \mathfrak{E}'$. Continuing this process, by Zorn's lemma L may be extended to some L^* defined on $\mathfrak{E}^*$.

THEOREM 1. *Let L be a positive linear functional defined on the linear manifold $\mathfrak{E} \subset C(\mathbf{R})$, and let $f_0 \in \mathfrak{E}$. If each function in $\mathfrak{E}$ is o-normal, then there exists at least one bounded monotone increasing function α on $\mathbf{R}$ such that*

$$L(f) = \int_{\mathbf{R}} f(u)\, d\alpha(u) \qquad (\text{all } f \in \mathfrak{E}). \tag{3.4}$$

Moreover, $[\operatorname{Var} \alpha]_{\mathbf{R}} = |L(f_0)| = |\alpha(\infty)|$.

Proof. Let $\{x_k\}_{k=-\infty}^{\infty}$ be the sequence of all rationals on $\mathbf{R}$ in any ordering. Define

$$\begin{aligned} \alpha(x_k) &= L(\chi_{(-\infty,x_k]}), \\ \alpha(x) &= \sup_{x_k<x} \alpha(x_k) \qquad (x \neq x_k). \end{aligned} \tag{3.5}$$

Since $L(\chi_{(-\infty,x_k]})$ is meaningful by Lemma 4 with $g = 1$, $A = (-\infty, x_k]$ (the extension of L being again denoted by L), the function α is well-defined on $\mathbf{R}$. Since $\chi_{(-\infty,x_k]} \leqslant \chi_{(-\infty,x_j]}$ for $x_k < x_j$, L being positive implies that α is monotone increasing on $\mathbf{R}$.

Now let f be any fixed element in $\mathfrak{C}$. Then for each $\epsilon > 0$ there exists an interval $J_0 = J_0(\epsilon)$ such that for all $J := [a, b] \supset J_0$ one has

$$-\epsilon\omega_f(u) \leqslant f(u) \leqslant \epsilon\omega_f(u) \qquad (\text{all } u \in \mathbf{R}\backslash[a, b]). \tag{3.6}$$

Let Δ be a partition of $[a, b]$ defined by $a = u_0 < u_1 < \cdots < u_p = b$, where the u_j are also rationals. One now applies Lemma 2, yielding that for each $\epsilon' > 0$ there is $\delta' = \delta'(\epsilon') > 0$ such that

$$-\epsilon' \leqslant f(u) - \sum_{k=1}^{p} f(u_k)\, \chi_{(u_{k-1},u_k]}(u) \leqslant \epsilon' \tag{3.7}$$

for all $u \in J$, $\| \Delta \| < \delta'$.

Since the $\chi_{(u_{k-1},u_k]}(u)$, $1 \leqslant k \leqslant p$, vanish for $u \in (a, b]$, inequalities (3.6), (3.7) may be combined to give

$$-\epsilon\omega_f(u) - \epsilon' \leqslant f(u) - \sum_{k=1}^{p} f(u_k)\, \chi_{(u_{k-1},u_k]}(u) \leqslant \epsilon' + \epsilon\omega_f(u) \tag{3.8}$$

for all $u \in \mathbf{R}$, $\| \Delta \| < \delta'$. Since

$$\chi_{(u_{k-1},u_k]} = \chi_{(-\infty,u_k]} - \chi_{(-\infty,u_{k-1}]},$$

and since L is linear and positive, it can be applied to inequalities (3.8) to yield, noting (3.5),

$$\begin{aligned} -\epsilon L(\omega_f) - \epsilon' L(f_0) &\leqslant L(f) - \sum_{k=1}^{p} f(u_k)[\alpha(u_k) - \alpha(u_{k-1})] \\ &\leqslant \epsilon' L(f_0) + \epsilon L(\omega_f) \qquad (\| \Delta \| < \delta'). \end{aligned} \tag{3.9}$$

It follows that one has, for $\|\Delta\| < \delta'(\epsilon')$ and any $\epsilon' > 0$,

$$\begin{aligned}\left| L(f) - \int_J f(u)\, d\alpha(u) \right| &\leqslant \left| L(f) - \sum_{k=1}^{p} f(u_k)[\alpha(u_k) - \alpha(u_{k-1})] \right| \\ &\quad + \left| \sum_{k=1}^{p} f(u_k)[\alpha(u_k) - \alpha(u_{k-1})] - \int_J f(u)\, d\alpha(u) \right| \\ &\leqslant \epsilon' L(f_0) + \epsilon L(\omega_f) + \epsilon',\end{aligned}$$

in view of (3.9) and the definition of the Riemann–Stieltjes integral for intervals J. Since ϵ' is arbitrary, one has

$$\left| L(f) - \int_J f(u)\, d\alpha(u) \right| \leqslant \epsilon L(\omega_f). \tag{3.10}$$

Letting $\epsilon \to 0$, which implies $J \to \mathbf{R}$ on account of (3.6), one deduces (3.4) for any $f \in \mathfrak{C}$. The rest of the proof is obvious.

Apart from the papers already mentioned [2, 5, 10], there are a number of other studies concerned with integral representations over $C(\mathbf{R})$, namely J. V. Wehausen [45], G. Sirvint [38], G. W. Mackey [29] and E. Hewitt [22]. But the approach of these papers is very different, generally more abstract; some treat $C(X)$, X a completely regular topological space (compactness absent). See also G. G. Gould and M. Mahowald [17] and J. D. Knowles [27].

Since the spaces $C(\mathbf{R})$ and P consist precisely of o-normal functions, applications of Theorem 1 yield

COROLLARY 1(a) *If L is a positive linear functional on $C(\mathbf{R})$, then there exists at least one increasing α on $\mathbf{R}$ such that the representation*

$$L(f) = \int_{\mathbf{R}} f(u)\, d\alpha(u) \tag{3.11}$$

holds for all $f \in C(\mathbf{R})$.

(b) *Each positive linear functional L on P admits the representation* (3.11) *for all $f \in P$ with an increasing α on $\mathbf{R}$.*

This corollary allows one to establish Theorem 2.

THEOREM 2 (Hamburger). *Let $\{\mu_n\}_{n=0}^{\infty}$ be an arbitrary sequence of reals. There exists at least one monotone increasing function α on $\mathbf{R}$ with infinitely many points of increase such that*

$$\mu_n = \int_{\mathbf{R}} u^n\, d\alpha(u) \qquad (n \in \mathbf{P}) \tag{3.12}$$

if and only if $\{\mu_n\}$ *is positive definite on* $\mathbf{R}$ (*i.e., for every polynomial*

$$p_l(x) = \sum_{k=0}^{l} a_k x^k$$

which is non-negative on $\mathbf{R}$ *one has* $\sum_{k=0}^{l} a_k \mu_k \geqslant 0$).

Proof. Let $\{\mu_n\}$ be positive definite. Given $p_l(x)$, set $L(p_l) = \sum_{k=0}^{l} a_k \mu_k$. Then L is a positive functional on P, linear by definition. By Corollary 2(b) there exists an increasing α on $\mathbf{R}$ such that (3.11) holds with f replaced by p_l, for any $l \in \mathbf{P}$. The particular choice $p_l(u) = u^n$, $n \in \mathbf{P}$, for which $L(u^n) = \mu_n$, yields (3.12). The converse is obvious.

Remark 1. Just as the problem of determining the general bounded linear functional on $C[0, 1]$ is equivalent to that of determining the set of all Hausdorff moment sequences, the question arises whether the Riesz theorem for $C(\mathbf{R})$ follows from Hamburger's theorem. However, Theorem 2 only yields Corollary 1(b), a representation on the subclass P of $C(\mathbf{R})$. It seems impossible to obtain one on all of $C(\mathbf{R})$ in this way, since P is not dense in $C(\mathbf{R})$. For other applications of Theorem 2, see Arens [5].

4. Representation Theorems for $C_0(\mathbf{R})$ and $C_{00}(\mathbf{R})$

Although each function of $C_0(\mathbf{R})$ is o-normal, Theorem 1 does not cover a representation theorem for $C_0(\mathbf{R})$, since the function $f_0(x) \equiv 1$ does not belong to $C_0(\mathbf{R})$. Therefore it must be established independently, and in fact for bounded linear functionals on $C_0(\mathbf{R})$.

Theorem 3. *If L is any bounded linear functional on $C_0(\mathbf{R})$, then there exists a unique $\alpha \in BV(\mathbf{R})$ such that*

$$L(f) = \int_{\mathbf{R}} f(u)\, d\alpha(u) \qquad (f \in C_0(\mathbf{R})). \tag{4.1}$$

Conversely, the right-hand member of (4.1) *defines a bounded linear functional on $C_0(\mathbf{R})$, and the norm of this functional is given by*

$$\| L \| = [\operatorname{Var} \alpha]_{\mathbf{R}}. \tag{4.2}$$

Proof. Let us first define a sequence of step-functions α_n on $\mathbf{R}$ as follows:

(i) $\alpha_n(0) = 0 \qquad (n \in \mathbf{N})$,

(ii) $\alpha_n\left(\frac{k}{n} + 0\right) - \alpha_n\left(\frac{k}{n} - 0\right) = L(S_{k,n})$, (4.3)

(iii) $\alpha_n(u) = \alpha_n\left(\frac{k}{n} + 0\right) \qquad \left(\frac{k}{n} \leqslant u < \frac{k+1}{n};\ k \in \mathbf{Z}\right).$

Then the variation of α_n over the interval $[-[R_1], [R_2]]$, $R_1, R_2 > 0$ ($[x]$ denoting the greatest integer $\leqslant x$) is given by

$$[\mathrm{Var}\,\alpha_n]_{-[R_1]}^{[R_2]} = \sum_{k=-n[R_1]+1}^{n[R_2]} \left|\alpha_n\left(\frac{k}{n}+0\right) - \alpha_n\left(\frac{k}{n}-0\right)\right|$$

$$= \sum_{k=-n[R_1]+1}^{n[R_2]} |L(S_{k,n})| = L\left(\sum_{k=-n[R_1]+1}^{n[R_2]} \epsilon_k S_{k,n}\right)$$

$$\leqslant \|L\| \left\| \sum_{k=-n[R_1]+1}^{n[R_2]} \epsilon_k S_{k,n} \right\|_C,$$

where $\epsilon_k = \mathrm{sign}\, L(S_{n,k})$.

Since one can readily show that

$$\left\| \sum_{k=-n[R_1]+1}^{n[R_2]} \epsilon_k S_{k,n} \right\| \leqslant 1$$

for any $n \in \mathbf{N}$, $R_1, R_2 > 0$, it follows that

$$[\mathrm{Var}\,\alpha_n]_{\mathbf{R}} \leqslant \|L\| \qquad (n \in \mathbf{N}). \tag{4.4}$$

Since clearly, for $f \in C_0(\mathbf{R})$,

$$\int_{-[R_1]}^{[R_2]} f(u)\, d\alpha_n(u) = \sum_{k=-n[R_1]+1}^{n[R_2]} f\left(\frac{k}{n}\right)\left[\alpha_n\left(\frac{k}{n}+0\right) - \alpha_n\left(\frac{k}{n}-0\right)\right],$$

one has on the one hand, by letting $R_1, R_2 \to \infty$, that

$$\int_{\mathbf{R}} f(u)\, d\alpha_n(u) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right)\left[\alpha_n\left(\frac{k}{n}+0\right) - \alpha_n\left(\frac{k}{n}-0\right)\right], \tag{4.5}$$

since the left integral exists and $\alpha_n \in BV(\mathbf{R})$.

On the other hand, one has for $f \in C_0(\mathbf{R})$, by Lemma 1, L being bounded, that

$$L(S_n f) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) L(S_{k,n}). \tag{4.6}$$

Combining (4.5) and (4.6) yields, by (4.3), that for $f \in C_0(\mathbf{R})$,

$$L(S_n f) = \int_{\mathbf{R}} f(u)\, d\alpha_n(u) \qquad (n \in \mathbf{N}). \tag{4.7}$$

In view of (4.4) the variations of α_n over $\mathbf{R}$ are uniformly bounded, and so, by the Helly–Bray theorem for $C_0(\mathbf{R})$, there exists a subsequence $\{\alpha_{n_k}\}_{k\in\mathbf{N}}$ and a function $\alpha \in BV(\mathbf{R})$ such that

$$\lim_{k\to\infty} L(S_{n_k}f) = \lim_{k\to\infty} \int_{\mathbf{R}} f(u)\, d\alpha_{n_k}(u) = \int_{\mathbf{R}} f(u)\, d\alpha(u). \tag{4.8}$$

But, by Lemma 1,

$$| L(S_{n_k}f) - L(f)| \leqslant \| L \| \, \| S_{n_k}f - f \|_C \to 0$$

as $k \to \infty$, so that the extreme left member of (4.8) is $L(f)$, yielding the desired representation (4.1), α being unique since it is normalized.

As to (4.2), $\| L \| \geqslant [\text{Var}\ \alpha]_{\mathbf{R}}$ is obvious by (4.4). The inverse inequality $\| L \| \leqslant [\text{Var}\ \alpha]_{\mathbf{R}}$ follows by (4.1). This completes the proof.

Concerning other literature pertaining to Theorem 3, W. Rudin [35, p. 131] and E. Hewitt and K. Stromberg [23, p. 363] establish it for $C_0(X)$, X being a locally compact Hausdorff space, via the representation theorem for $C_{00}(X)$ (plus the Radon–Nikodym theorem with an $L^1(X)$-representation theorem). (See also R. F. Arens [4] and Z. Semadeni [36, p. 312]). We, on the contrary, deduce now the result for $C_{00}(\mathbf{R})$ as a direct application of Theorem 3.

COROLLARY 2. *If L is any bounded linear functional on $C_{00}(\mathbf{R})$, then there exists a unique $\alpha \in BV(\mathbf{R})$ such that*

$$L(f) = \int_{\mathbf{R}} f(u)\, d\alpha(u) \qquad (f \in C_{00}(\mathbf{R})). \tag{4.9}$$

Conversely, the integral in (4.9) *defines a bounded linear functional on $C_{00}(\mathbf{R})$ whose norm is given by* (4.2).

Proof. Since $C_{00}(\mathbf{R})$ is dense in $C_0(\mathbf{R})$, one can extend L uniquely to $C_0(\mathbf{R})$ while preserving the norm. The result now follows by Theorem 3.

The literature on $C_{00}(\mathbf{R})$ representation theorems is very abundant, e.g. E. Asplund and L. Bungart [6, p. 362, 372], A. E. Taylor [40, p. 374], R. F. Arens [4] and H. L. Royden [34, p. 251], [44].

5. REPRESENTATION THEOREMS FOR $C_B(\mathbf{R})$ AND $B(\mathbf{R})$

The next question is whether the most general bounded or even positive, linear functional L on $C_B(\mathbf{R})$ is also expressible as a simple Riemann–Stieltjes integral

$$L(f) = \int_{\mathbf{R}} f(u)\, d\alpha(u) \qquad (f \in C_B(\mathbf{R})), \tag{5.1}$$

with $\alpha \in BV(\mathbf{R})$ or α monotone increasing on $\mathbf{R}$. Generally the answer is negative as noted in several places (e.g., [25, p. 99], [12, p. 240], [10, p. 294]). But since the authors have seen no explicit proof of this fact, it will be now established, for the sake of completeness. First note that positivity is a stronger condition upon L defined on the normed linear space $C_B(\mathbf{R})$ than boundedness: if L is positive, then $|L(f)| \leqslant \|f\|_C L(f_0)$, implying that $\|L\| \leqslant L(f_0) < \infty$.

LEMMA 5. (a) *There is at least one positive (and therefore bounded) linear functional* L *on* $C_B(\mathbf{R})$ *such that there does not exist a monotone increasing function* α *on* $\mathbf{R}$ *with* $[\operatorname{Var} \alpha]_{\mathbf{R}} < +\infty$ *for which* $L(f) = \int_{\mathbf{R}} f(u)\, d\alpha(u)$ *for every* $f \in C_B(\mathbf{R})$.

(b) *The same negative result is valid for bounded linear functionals* L *on* $B(\mathbf{R})$.

Proof. (a) Beginning with an idea of Hewitt [22, p. 271], set

$$p(f) = \lim_{x\to\infty} \sup f(x) \qquad (f \in C_B(\mathbf{R})). \tag{5.2}$$

Then $p(f+g) \leqslant p(f) + p(g)$ for all $f \in C_B(\mathbf{R})$, and $p(\alpha f) = \alpha p(f)$ for $\alpha \geqslant 0$. On the subspace M of $C_B(\mathbf{R})$ for which $\lim_{x\to\infty} f(x)$ exists, p is a positive linear functional. By the Hahn–Banach extension theorem there exists a positive, linear functional L on $C_B(\mathbf{R})$ such that

$$\begin{aligned} -p(-f) \leqslant L(f) \leqslant p(f) &\qquad (f \in C_B(\mathbf{R})), \\ L(f) = p(f) &\qquad (f \in M). \end{aligned} \tag{5.3}$$

Now assume that a representation (5.1) does hold for all $f \in C_B(\mathbf{R})$, with a monotone increasing α on $\mathbf{R}$ satisfying $[\operatorname{Var} \alpha]_{\mathbf{R}} < +\infty$. Then, for the particular $f_0 \equiv 1 \in M \subset C_B(\mathbf{R})$, one has $L(f_0) = \int_{\mathbf{R}} d\alpha(u)$. But, on the other hand, $L(f_0) = p(f_0) = \lim_{x\to\infty} f_0(x) = 1$ by (5.3), giving

$$1 = \int_{\mathbf{R}} d\alpha(u) = \alpha(\infty) - \alpha(-\infty). \tag{5.4}$$

Since $L(g) = \lim_{x\to\infty} g(x)$ for $g \in M$ by (5.2) and (5.3), $L(g) = 0$ for $g \in C_0(\mathbf{R})$ since $C_0(\mathbf{R}) \subset M$. Thus

$$\int_{\mathbf{R}} g(u)\, d\alpha(u) = 0 \qquad (g \in C_0(\mathbf{R})).$$

But this readily implies that $\alpha = \text{const}$ on $\mathbf{R}$, and so $\alpha(\infty) - \alpha(-\infty) = 0$, which contradicts (5.4).

(b) Let L be the positive (and so bounded) linear functional on $C_B(\mathbf{R})$, whose existence was stated in Lemma 5(a). By the Hahn–Banach theorem, L may be extended to a bounded linear functional on $B(\mathbf{R})$ (again denoted by L). Now if (5.1) were valid for each bounded linear functional on $B(\mathbf{R})$, then it would also be valid on $C_B(\mathbf{R})$, which contradicts part (a).

In view of these negative results, can one still represent the positive linear functionals on $C_B(\mathbf{R})$ as some generalized integral?

Following, for example, Taylor [41, p. 401; 47; 21; 11, p. 95f], we have the following definition

DEFINITION 3. Let Ω be any nonempty set and $\mathfrak{P}(\Omega)$ the set of all subsets of Ω. A finitely additive measure (or charge) μ on $\mathfrak{P}(\Omega)$ is a mapping of $\mathfrak{P}(\Omega)$ into $\mathbf{R}$ such that $\mu(\varnothing) = 0$ ($\varnothing$ being the empty set), $\mu(A \cup B) = \mu(A) + \mu(B)$ for all $A, B \in \mathfrak{P}(\Omega)$ with $A \cap B = \varnothing$, and $\sup_{A \in \mathfrak{P}(\Omega)} |\mu(A)| < +\infty$. If $\mu \geqslant 0$, then the measure is called *positive*.

Remark 2. If L is a positive linear functional on $C_B(\mathbf{R})$, then L can be extended to a positive functional defined on $C_B(\mathbf{R}) + \text{span } X(C_B(\mathbf{R}))$, by Lemma 4, so that $L(\chi_E)$ is defined for all $E \subset \mathbf{R}$. Then

$$\mu(E) := L(\chi_E)$$

defines a finitely additive positive measure on $\mathfrak{P}(\Omega)$. Indeed, $\mu(\varnothing) = L(\chi_\varnothing) = L(0) = 0$, and $\mu(A \cup B) = L(\chi_{A \cup B}) = L(\chi_A + \chi_B) = \mu(A) + \mu(B)$ for all $A, B \in \mathfrak{P}(\Omega)$, $A \cap B = \varnothing$. Moreover, $|\mu(A)| = |L(\chi_A)| \leqslant \|L\| \|\chi_A\| = \|L\|$ for all $A \in \mathfrak{P}(\Omega)$, so that $\sup |\mu(A)| < +\infty$. As L is positive, $\mu \geqslant 0$, since $\chi_A \geqslant 0$ for all $A \in \mathfrak{P}(\Omega)$.

Of the possible definitions of an integral for finitely additive measures[1] we select the following (compare [22; 41, p. 401; 24]):

DEFINITION 4. Let μ be a finitely additive measure on $\mathfrak{P}(\Omega)$, and let f be a bounded scalar-valued function defined on Ω. Let c, C, d, D and Δ' be defined as in Lemma 3. If the expression

$$\lim_{\|\Delta'\| \to 0} \sum_{k=1}^{p} \beta_{k-1} \mu(f^{-1}(\beta_{k-1}, \beta_k])$$

exists independently of the choice of Δ', then it is called the μ-integral of f on Ω, and is denoted by $\int_\Omega f \, d\mu$.

[1] In contrast to Edwards [12, p. 213], H. Günzler in his lecture notes on "Integration" (mimeographed, Univ. of Kiel, 1971, p. 31) speaks of such an integral as a proper (abstract) Riemann integral.

Properties of the μ-Integral.

(i) It is easy to verify that this integral is a linear functional on the classes $C_B(\mathbf{R})$, $B(\mathbf{R})$ for the choice $\Omega = \mathbf{R}$.

(ii) The μ-integral is positive provided μ is a positive finitely additive measure.

(iii) For $A \in \mathfrak{P}(\mathbf{R})$, let $\mathfrak{A}$ denote a finite collection of pairwise disjoint sets $A_1, \ldots, A_p$ from $\mathfrak{P}(\mathbf{R})$ such that $A_k \subset A$. Set

$$\|\mu\| := \sup_{A \in \mathfrak{P}(\mathbf{R})} \left(\sup_{\mathfrak{A}} \sum_{k=1}^{p} |\mu(A_k)| \right).$$

Since μ is finite by Definition 3, $\|\mu\|$ is also finite, and one has for every $f \in C_B(\mathbf{R})$,

$$\left| \int_{\mathbf{R}} f \, d\mu \right| \leqslant \|f\|_C \|\mu\|.$$

Although the following result (announced above) is essentially contained in Hewitt [22], we present it here for completeness, with a different proof (which also makes use of Lemma 4).

THEOREM 4. *For each positive linear functional L on $C_B(\mathbf{R})$ there exists a finitely additive positive measure μ defined on $\mathfrak{P}(\mathbf{R})$ such that*

$$L(f) = \int_{\mathbf{R}} f \, d\mu \qquad (f \in C_B(\mathbf{R})). \tag{5.5}$$

Conversely, the right side of (5.5) *defines a positive linear functional on $C_B(\mathbf{R})$, whose norm is given by*

$$\|L\| = \|\mu\|. \tag{5.6}$$

Proof. Since f is bounded, $c = \inf f(x)$, $d = \sup f(x)$ are both finite. Using the notation of Lemma 3, for each $\epsilon > 0$ there is $\delta(\epsilon) > 0$ such that

$$-\epsilon \leqslant f(u) - \sum_{k=1}^{p} \beta_{k-1} \chi_{E_k}(u) \leqslant \epsilon \qquad (\|\Delta'\| < \delta;\ u \in \mathbf{R})$$

with $E_k = f^{-1}(\beta_{k-1}, \beta_k]$, $1 \leqslant k \leqslant p$. Applying L to this inequality, L being positive and linear, gives

$$-\epsilon L(f_0) \leqslant L(f) - \sum_{k=1}^{p} \beta_{k-1} L(\chi_{E_k}) \leqslant \epsilon L(f_0) \quad (\|\Delta'\| < \delta).$$

Defining a positive finitely additive measure μ by

$$\mu(E) := L(\chi_E) \qquad (E \in \mathfrak{P}(\mathbf{R})), \tag{5.7}$$

as pointed out in Remark 2 (which made use of Lemma 4), one has

$$\left| L(f) - \sum_{k=1}^{p} \beta_{k-1}\mu(E_k) \right| < \epsilon L(f_0) \quad (\| \Delta' \| < \delta).$$

Since Δ' is arbitrary, the representation (5.5) follows by Definition 4.

Concerning (5.6), property (iii) of the μ-integral yields $\| L \| \leqslant \| \mu \|$. Conversely, $\| \mu \| \leqslant \| L \|$ by (5.7).

THEOREM 5. *For each bounded linear functional on* $C_B(\mathbf{R})$, *there exists a finitely additive measure* μ *on* $\mathfrak{P}(\mathbf{R})$ *such that* (5.5) *and* (5.6) *hold.*

Proof. Since L is a bounded linear functional on $C_B(\mathbf{R})$, it can be extended to a bounded linear functional on $B(\mathbf{R})$ by the Hahn–Banach theorem, so that $\mu(E) := L(\chi_E)$, $E \in \mathfrak{P}(\mathbf{R})$, is again well-defined (the extension of L being again denoted by L). Using the notations of Lemma 3 one has, L being linear and bounded,

$$\left| L(f) - \sum_{k=1}^{p} \beta_{k-1}\mu(E_k) \right| \leqslant \| L \| \left\| f - \sum_{k=1}^{p} \beta_{k-1}\chi_{E_k} \right\|_C,$$

which tends to zero as $\| \Delta' \| \to 0$, by Lemma 3. This gives the result, since (5.6) follows as before.

Concerning the literature, Riesz-type theorems for linear functionals on $C_B(\mathbf{R})$ were apparently first studied by G. Fichtenholz and L. Kantorovitch [13] in 1934, then by A. A. Markoff [30], A. D. Alexandroff [3] and E. Hewitt [22, p. 280]. See also P. C. Rosenbloom [33], G. G. Gould [16] and D. Fremlin, D. Garling and R. Haydon [14].

COROLLARY 3. *If L is any bounded linear functional defined on* $B(\mathbf{R})$, *then there exists a finitely additive measure* μ *on* $\mathfrak{P}(\mathbf{R})$ *such that* (5.5) *holds for all* $f \in B(\mathbf{R})$, *as well as* (5.6).

Proof. The proof is analogous to that of Theorem 5 but simpler since the extension argument is superfluous as $\chi_E \in B(\mathbf{R})$ for all $E \in \mathfrak{P}(\mathbf{R})$.

Riesz-type theorems for $B(\mathbf{R})$ were first studied by T. H. Hildebrandt [24] in 1934. See also E. Hewitt [20], I. Glicksberg [15] and especially A. E. Taylor [41, p. 403].

Finally, is it possible to obtain a representation of L for positive functionals on $C_B(\mathbf{R})$ which is more concrete than (5.5) provided f is restricted somewhat. Indeed, already as an application of Theorem 1 we have the following.

Each positive linear functional L on $C_B(\mathbf{R})$ admits the representation

$$L(f) = \int_{\mathbf{R}} f(u)\, d\alpha(u) \tag{5.8}$$

on the subspace of $C_B(\mathbf{R})$ consisting of the o-normal functions.

However, as seen above, a representation of L valid on its whole domain of definition $C_B(\mathbf{R})$ is possible with finitely additive measures. Concerning a more substantial representation than (5.5) but for bounded linear functionals on $C_U(\mathbf{R})$, the subset of these $f \in C(\mathbf{R})$ which are uniformly continuous on $\mathbf{R}$, we have via Theorem 3, Corollary 4.

COROLLARY 4. *For each bounded linear functional L on $C_U(\mathbf{R})$ there exists a sequence $\{\alpha_n\}_{n=0}^{\infty}$ of step-functions belonging to $BV(\mathbf{R})$ for which*

$$[\mathrm{Var}\ \alpha_n]_{\mathbf{R}} \leqslant \| L \|, \quad n \in \mathbf{N}$$

and such that

$$L(f) = \lim_{n\to\infty} \int_{\mathbf{R}} f(u)\, d\alpha_n(u) \quad (f \in C_U(\mathbf{R})), \tag{5.9}$$

$$\| L \| = \lim_{n\to\infty} \sup\ [\mathrm{Var}\ \alpha_n]_{\mathbf{R}}\,. \tag{5.10}$$

The proof proceeds as for Theorem 3 up to the stage of formula (4.7). Here the left member tends to $L(f)$ by Lemma 1 (also valid when $C_0(\mathbf{R})$ is replaced by $C_U(\mathbf{R})$), but the right member does not converge to $\int_{\mathbf{R}} f(u)\, d\alpha(u)$ since the Helly–Bray theorem does not hold for $C_U(\mathbf{R})$.

Concerning (5.10): that $\lim\sup_{n\to\infty} [\mathrm{Var}\ \alpha_n]_{\mathbf{R}} \leqslant \| L \|$ follows by (4.4). The converse follows by noting that (5.9) implies.

$$\begin{aligned} |L(f)| &= \lim_{n\to\infty} \left| \int_{\mathbf{R}} f(u)\, d\alpha_n(u) \right| \\ &\leqslant \lim_{n\to\infty} \sup \|f\| \cdot [\mathrm{Var}\ \alpha_n]_{\mathbf{R}}\,. \end{aligned}$$

This corollary improves a result of Hildebrandt [24].

As to further literature, there is N. Bourbaki [8, Chapter III, pp. 41–102] as well as recent work by F. Topsøe [42, 43] on a unified approach to representation theorems covering the spaces $C_B(X)$, $C_{00}(X)$, $C(X)$, X being, e.g., locally compact. For very recent work see especially the general approach of H. Günzler [18a, b, c] which takes care of the spaces $C_{00}(X)$, $C_0(X)$, $C(X)$ $C_B(X)$, X being *any* topological space.

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